

THEORETICAL AMPLITUDES OF THE SEISMIC PHASE *PKJKP*

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Summary

Calculations have been made of the partitioning of energy of *P* waves incident from above against the boundary of the Earth's inner core, and of *P* and *SV* waves incident against this boundary from below. In these calculations the inner core has been assumed to have rigidity as in an Earth model based on compressibility theory. The calculations have been applied to estimating the relative amplitudes of the phases *PKJKP* and *PKIKP* at various epicentral distances for a given earthquake. It is shown that, on the assumptions made, the phase *PKJKP* is most likely to be observed over a range $130^\circ \leq \Delta \leq 155^\circ$ of epicentral distance, the amplitude in this range being about one-fifth of that for *PKIKP*. It is suggested that readers of seismograms who find *PKIKP* strongly recorded in this range might also look for the phase *PKJKP*; the writer of this paper would much appreciate it if observational evidence supporting the existence of the phase *PKJKP* could be communicated to him for purposes of collation.

Theoretical travel-time tables have been constructed * for the phase *PKJKP*, where *J* corresponds to a transit of the Earth's inner core in *S* type. The purpose of the present paper is to compute the expected amplitudes of such a phase over the relevant range of angular distance.

By the time waves have reached the inner core they will have been subjected to refraction at several discontinuity surfaces. Thus calculations of the absolute distribution of the amplitude of *PKJKP* would be exceedingly complicated, and moreover subject to high uncertainty through lack of precise knowledge of transmission losses in the regions outside the inner core, especially in the crustal layers. The best progress can therefore be made by determining the amplitude of a *PKJKP* phase for given distance as a fraction of the amplitude of the *PKIKP* phase which has the same parameter *p*; for the rays corresponding to these two phases have similar paths outside the inner core.

The symbols *e*, *f* will denote the angles of emergence in the region, *M* say, just inside the inner core, of *P* and *SV* rays whose parameter is *p*; *e'* will similarly apply to a *P* ray in the region, *M'* say, just outside the inner core, the parameter *p* being the same.

Consider first a *P* wave incident against the inner core from above. In standard notation (see B†, pp. 86, 100–101), partitioning of the waves (treated as plane waves) at the boundary of the inner core is described by the equations

$$\phi = A'_0 \exp \{i\kappa(z \tan e' + x - ct)\} + A' \exp \{i\kappa(-z \tan e' + x - ct)\},$$

$$\psi = 0,$$

holding in *M'*, and

$$\phi = A \exp \{i\kappa(z \tan e + x - ct)\}, \quad (1)$$

$$\psi = B \exp \{i\kappa(z \tan f + x - ct)\}, \quad (2)$$

* K. E. Bullen, "Theoretical Travel-times of *S* waves in the Earth's Inner Core", *M.N. Geophys. Suppl.*, 6, 125, 1950; this paper will be subsequently referred to as Paper I.

† K. E. Bullen, *Theory of Seismology*, Cambridge University Press, 1947.

holding in M . In these equations, ϕ represents P waves and ψ SV waves; the x -axis is in the boundary (curvature of which is here neglected) and is perpendicular to lines in which wave-fronts cut the boundary, and the z -axis is normal to the boundary and pointing downward. The displacement components u_1, u_3 parallel to the x -, z -axes are given by $\partial\phi/\partial x + \partial\psi/\partial z$, $\partial\phi/\partial z - \partial\psi/\partial x$, respectively.

It is sufficient for our purpose to determine the ratio B/A , and this comes immediately from the boundary condition which requires the stress-component p_{zx} to be zero at the outer boundary of the inner core because of the fluidity of the material above. From the formula (**B**, p. 88)

$$p_{zx} = \mu \left(2 \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right),$$

we immediately derive by (1) and (2)

$$\left| \frac{B}{A} \right| = \frac{2 \tan e}{|\tan^2 f - 1|}. \quad (3)$$

For an assigned small range of values of p , the energies associated with (1), (2) are proportional to $A^2 \rho \tan e$, $B^2 \rho \tan f$, respectively, where ρ denotes the density in M . Hence, by (3), the energies in S and P types refracted into the inner core are in the ratio

$$\frac{4 \tan e \tan f}{(\tan^2 f - 1)^2}. \quad (4)$$

Next, consider the P waves emerging at angle e against the inner core boundary from below. The reflection-refraction equations are now of the form

$$\phi = A_0 \exp \{i\kappa(z \tan e + x - ct)\} + A_1 \exp \{i\kappa(-z \tan e + x - ct)\},$$

$$\psi = B_1 \exp \{i\kappa(-z \tan f + x - ct)\}$$

in M , and

$$\phi = A'_1 \exp \{i\kappa(z \tan e' + x - ct)\},$$

$$\psi = 0$$

in M' . The relevant boundary conditions are the continuity of u_3 , p_{zx} , p_{zz} across the boundary. These lead to the equations

$$(A_1 - A_0) \tan e + A'_1 \tan e' + B_1 = 0, \quad (5)$$

$$2(A_1 - A_0) \tan e + B_1(1 - \tan^2 f) = 0, \quad (6)$$

$$(A_1 + A_0) \cos 2f + A'_1 \rho' / \rho - B_1 \sin 2f = 0, \quad (7)$$

where ρ' is the density in M' .

For SV waves incident at angle f against the inner core boundary from below, the seven equations of the last paragraphs are replaced by the forms

$$\begin{cases} \phi = A_2 \exp \{i\kappa(-z \tan e + x - ct)\}, \\ \psi = B_0 \exp \{i\kappa(z \tan f + x - ct)\} + B_2 \exp \{i\kappa(-z \tan f + x - ct)\}, \\ \phi = A'_2 \exp \{i\kappa(z \tan e' + x - ct)\}, \\ \psi = 0, \end{cases}$$

$$\begin{cases} A_2 \tan e + A'_2 \tan e' + (B_2 + B_0) = 0, \end{cases} \quad (8)$$

$$\begin{cases} 2A_2 \tan e + (B_2 + B_0)(1 - \tan^2 f) = 0, \end{cases} \quad (9)$$

$$\begin{cases} A_2 \cos 2f + A'_2 \rho' / \rho - (B_2 - B_0) \sin 2f = 0. \end{cases} \quad (10)$$

From (5), (6), (8) and (9), we derive

$$\frac{A_2}{A_1 - A_0} = \frac{B_2 + B_0}{B_1} = \frac{A'_2}{A'_1} = h, \text{ say.} \quad (11)$$

Substituting into (10), we have

$$(A_1 - A_0)h \cos 2f + A'_1 h \rho' / \rho - (B_1 h - 2B_0) \sin 2f = 0. \quad (12)$$

By (7) and (12),

$$2A_0 h \cos 2f - 2B_0 \sin 2f = 0,$$

so that

$$h = (B_0 / A_0) \tan 2f.$$

Thus, by (11),

$$\left(\frac{A'_2}{B_0} \right) / \left(\frac{A'_1}{A_0} \right) = \tan 2f. \quad (13)$$

Now the energies associated with the terms involving A_0 , B_0 , A'_1 , A'_2 in the expressions for ϕ , ψ are proportional to $A_0^2 \rho \tan e$, $B_0^2 \rho \tan f$, $A_1'^2 \rho' \tan e'$, $A_2'^2 \rho' \tan e'$, respectively. Thus in comparing the energies that go into *KJK*, *KIK* waves from a given *K*-pulse incident against the inner core, we have, using (13), to apply in addition to (4) the factor

$$\frac{\tan e \tan^2 2f}{\tan f} \quad (14)$$

in order to allow for energy changes at the second refraction. The product, a^2 say, of (4) and (14) is given by

$$a = \frac{2 \tan e \tan 2f}{|\tan^2 f - 1|}. \quad (15)$$

Further allowance has theoretically to be made for the different spreads of the wave-front areas of the *I* and *J* waves for a given range of p . If Δ_1 , Δ_2 are the total distances for *PKIKP*, *PKJKP* for given p , the further energy factor, b^2 say, is seen (using **B**, p. 125, equation (4)) to be given by

$$b^2 = \frac{\left(\frac{dp}{d\Delta} \right)_2 \sin \Delta_1}{\left(\frac{dp}{d\Delta} \right)_1 \sin \Delta_2}. \quad (16)$$

Neglecting possible absorption inside the inner core, it follows that the amplitude of *PKJKP* at Δ_2 will be equal to ab times that of *PKIKP* at Δ_1 , where a , b are given by (15), (16).

For any given value of p , the values of e and f are calculated from the formulae

$$\cos e, \cos f = 57 \cdot 3 (\alpha, \beta) p / r,$$

where α , β are the *P*, *S* velocities in *M*, r is the radius of the inner core, and the units of p are sec./deg. Taking α equal to 11.16 km./sec., and β equal to 0.433 α , corresponding to the main calculations in Paper I, values of e and f are found as shown in Table I below. In Table I, Δ_1 and Δ_2 respectively denote the distances of *PKIKP* and *PKJKP* rays corresponding to the values of p . Values of Δ_1 are derived from the J.B. tables of *PKIKP*, and of Δ_2 from Table III of Paper I. In the last column, values of the factor a given by (15) are shown.

TABLE I
Elements in the comparison of amplitudes of *PKIKP*, *PKJKP*

p	e	f	Δ_1	Δ_2	a
0.00	90°	90°	180°	180°	0.00
0.28	82°	86°	175°	185°	0.01
0.55	74°	83°	171°	190°	0.03
0.80	66°	80°	166°	195°	0.05
1.03	58°	77°	162°	200°	0.09
1.23	51°	74°	158°	205°	0.13
1.41	44°	72°	153°	210°	0.17
1.56	37°	70°	147°	215°	0.20
1.67	31°	68°	141°	220°	0.21
1.77	25°	67°	136°	225°	0.23
1.86	16°	65°	131°	230°	0.19
1.93	5°	64°	126°	235°	0.05

From Table I, the very useful result emerges that $\Delta_1 + \Delta_2 = 360^\circ$ within an error of at most 3° . Likewise $(dp/d\Delta)_1$ and $(dp/d\Delta)_2$ are equal, to sufficient accuracy, so that the factor b can be taken as unity. Hence the values of a given in Table I give the theoretical ratios of the amplitude of *PKJKP* at Δ_2 to that of *PKIKP* at Δ_1 .

The fact that $\Delta_1 + \Delta_2 \approx 360^\circ$ is, further, of major importance in facilitating the search for the phase *PKJKP*. It means that on the theory presented in this paper, the appropriate value of a gives the expected amplitude ratio of *PKJKP* and *PKIKP* phases arriving at one and the same epicentral distance.

It is therefore suggested that when a strong *PKIKP* phase is recorded on a seismogram, a search be made on the same seismogram for a pulse *PKJKP* arriving at a time given in Table III of Paper I (or possibly earlier, if there is a density jump across the boundary of the inner core) and with amplitude a times that of the *PKIKP*. *The writer of this paper would be very grateful if significant details of arrival-times and amplitudes of any relevant observations were communicated to him.* A crucial feature in determining the reality of the *PKJKP* phase would be the gradient of an empirical time-distance curve, and a large number of observations would be needed to establish the existence of *PKJKP* and to determine the gradient with precision.

It is seen from Table I that the expected amplitude of *PKJKP* as given by the coefficient a is about one-fifth of the corresponding amplitude of *PKIKP* if the epicentral distance is approximately in the range $130^\circ \leq \Delta \leq 150^\circ$. The ratio drops to about one-tenth at $\Delta = 160^\circ$, and is one-twentieth or less for $\Delta < 125^\circ$, or $\Delta > 165^\circ$. Thus, provided the amplitude of *PKIKP* is fairly steady over the range of distance in which it is observed, the phase *PKJKP* is most likely to be observed at stations whose epicentral distances are in or near the range $130^\circ \leq \Delta \leq 155^\circ$.

Useful data on the amplitudes of the phase P' have been prepared by Denson*, and Professor Gutenberg has kindly sent me photographed copies of graphs summarizing the data. In the graphs, points are plotted for P' over the range $110^\circ \leq \Delta \leq 180^\circ$ giving logarithms of components of the ground amplitude divided by the period. Separate graphs are given for (i) long-period horizontal instruments, (ii) short-period vertical instruments, (iii) long-period vertical instruments.

* M. E. Denson, Jr., "Longitudinal Waves through the Earth's Core", Ph.D. Thesis, California Institute of Technology, 1950.

TABLE II
Variation of amplitude/period for *P'*

Δ	(i)	(ii)	(iii)	(iv)	(v)
112.5	1	2	1	2	2
117.5	1	3	1	3	2
122.5	...	4	3	4	4
127.5	2	10	3	6	5
132.5	2	13	5	8	6
137.5	2	10	6	8	5
142.5	3	8	10	8	6
147.5	5	40	13	20	16
152.5	3	30	10	16	13
157.5	2	16	9	10	5
162.5	2	20	4	8	5
167.5	2	25	8	13	6
172.5	2	8	8	8	6

From Dr Denson's graphs I have determined mean values over 5° ranges centred at 112.5 , 117.5 , etc., and then computed the corresponding quotients of amplitude/period in arbitrary units. The results are shown in Table II where the columns (i), (ii), (iii) correspond to (i), (ii), (iii) of the preceding paragraph; (iv) to the weighted mean of (ii) and (iii); and (v) to the weighted mean of (i), (ii), (iii). The standard deviation is high in a number of entries, both because the number of observations is small and because the dispersion is considerable; thus only the general trend of the figures in Table II is significant.

The table implies that over the range $125^\circ \leq \Delta \leq 180^\circ$ the amplitude of *P'* is fairly steady, except for the entries at $\Delta = 147.5$, 152.5 . These larger amplitudes are, however, likely to be associated with the branch *ABC** of the *P'* curve, and not with the branch *DF* which relates to *PKIKP*. The work of Denson therefore is compatible with the conclusion that *PKJKP* is most likely to be observed in or near the range $130^\circ \leq \Delta \leq 155^\circ$.

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* H. Jeffreys and K. E. Bullen, *Seismological Tables*, British Association, Fig. 2, page 6, 1940.